Question

1. ***Problem 1: (Poisson Process and Exponential Distribution)***
   Consider K independent sources of customers where the inter-arrival time between customers for each source is exponential distributed with parameter λ_k (i.e., each source is a Poisson process). Now consider the arrival stream, which is formed by merging the input from each of the K sources defined above. Prove that this stream is also Poisson with parameter $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_K$.

2. ***Problem 2: (Waiting Time)***
   A barber opens up for business at $t = 0$. Customers arrivar at random in a Poisson fashion: that is, the pdf of the interarrival time is $a(t) = \lambda e^{-\lambda t}$. Each haircut takes X secs (where X is some random variable). Find the probability P that the second arriving customer will not have to wait and also find W, the average value of this waiting time for the two following cases:

   (a) $X = c =$ constant.
   (b) X is exponrntial distributed with pdf: $b(x) = ue^{-ux}$.

3. ***Problem 3: (Discrete State, Continuous Time Markov Chain)***
   Consider a pure Markovian queueing system in which

   $$\begin{array}{ll} \lambda_k = \lambda & 0 \leq k \leq K \\ \lambda_k = 2\lambda & K < k \\ \mu_k = \mu & k = 1, 2, \cdots \end{array}$$

   Find the equilibrium probabilities p_k for the number in the system.

4. ***Problem 4: (Discrete State, Continuous Time Markov Chain)***
   Consider a M/M/2 queueing system where the average arrival rate is λ customer per second and the average service time is $1/\mu$ sec. where $\lambda < 2\mu$.
   (a) Find the differential equations that govern the time-dependent probability $P_k(t)$.
   (b) Find the equilibrium probabilities $p_k = lim_{i \to \infty} P_k(t)$.

5. ***Problem 5: (Discrete State, Continuous Time Markov Chain)***

   Consider a Markovian queueing system in which

   $$\begin{array}{ll} \lambda_k = \alpha^k \lambda & k \geq 0, 0 \leq \alpha < 1 \\ \mu_k = \mu & k = 1, 2, \cdots \end{array}$$

(a) Find the equilibrium probability p_k of having k customers in the system. Express your answer in terms of p_0.
(b) Give an expression for p_0.

6. ***Problem 6: (Discrete State, Continuous Time Markov Chain)***
Here we consider an M/M/a queue in discrete time where time is segmented into intervals of length q sec each. We assume that events can only occur at the ends of these discrete time intervals. In particular the probability of a single arrival at the end of such an interval is given by λq and the probability of no arrival at that point is $1-\lambda q$ (thus at most one interval may occur). Similarly the departure process is such that if a customer is in service during an interval he will complete service at the end of that interval with probability $1-\sigma$ or will require at least no more interval with probability σ.
(a) Derive the form of a(t) and b(x), the inter-arrival time and service time pdf's, respectively.
(b) Assuming FCFS, write down the equilibrium equations that govern the behavior of p=$Pr\{k$ customers in system at the end of a discrete interval$\}$, where k includes any arrival who have occurred at the end of this interval as well as any customers who are about to leave at this point.
(c) Solve for the expected value of the number of customers at these points.

Answer

1. The moment generating function of the i_{st} source (Poisson random variable X_i) is $\phi_{X_i}(t)$, in which
$$\phi_{X_i}(t) = e^{\lambda_i(e^t-1)}.$$
Let $Y = X_1 + X_2 + \cdots + X_K$ is the random variable which is the merging stream of K sources. Therefore,
$$\begin{aligned}\phi_Y(t) &= \phi_{X_1+\cdots+X_K}(t)\\ &= \phi_{X_1}(t)\phi_{X_2}(t)\cdots\phi_{X_K}(t)\\ &= e^{\lambda_1(e^t-1)}e^{\lambda_2(e^t-1)}\cdots e^{\lambda_K(e^t-1)}\\ &= e^{(\lambda_1+\cdots+\lambda_K)(e^t-1)}.\end{aligned}$$
Therefore, Y is the Poisson random variable with parameter $\lambda = \lambda_1 + \lambda_2 + \cdots + \lambda_K$.

2. The probability that the second customer will not have to wait is P=Pr{inter-arrival time $Y \geq$ the service time of first customer X}

(a)

P=$Pr\{Y \geq c\}$: that is
$$P = 1 - (1 - e^{-\lambda c}) = e^{-\lambda c}.$$
$$\begin{aligned}W &= \int_{t=0}^{c}(c-t)\lambda e^{-\lambda t}dt\\ &= c(1-e^{-\lambda t}) - 1/\lambda[1 - e^{-\lambda c}(\lambda c + 1)].\end{aligned} \tag{1}$$

(b)

P=$Pr\{Y \geq X\}$; that is
$$\begin{aligned}P &= \int_{x=0}^{\infty} Pr\{Y \geq X|X=x\}f_X(x)dx\\ &= \int_{x=0}^{\infty} Pr\{Y \geq x\}\lambda e^{-ux}dx\\ &= \int_{x=0}^{\infty} e^{-\lambda x}ue^{-ux}dx\\ &= \int_{x=0}^{\infty} ue^{-(\lambda+u)x}dx\\ &= \frac{u}{\lambda+u}.\end{aligned}$$

$$W = [(x-t)P(X \le Y \mid Y = x)]P(Y = x)$$
$$= \int_0^\infty \int_0^x [(x-t)\lambda e^{-\lambda t}]\mu e^{-\mu x} dt dx$$
$$= \int_0^\infty \left(x - \frac{1}{\lambda}(1 - e^{-\lambda x}) \right) \mu e^{-\mu x} dx$$
$$= \left[e^{-\mu x} \left(-\frac{x}{\mu} - \frac{1}{\mu^2} + \frac{1}{\lambda} \right) - \frac{\mu}{\lambda + \mu} e^{-(\lambda+\mu)x} \right]_0^\infty$$

3. The balanced equation is

$$\begin{array}{ll} \lambda p_0 = \mu p_1 & \\ \lambda p_k = \mu p_{k+1} & 1 \le k \le K \\ 2\lambda p_k = \mu p_{k+1} & k \ge K+1 \\ \sum_{i=0}^{\infty} p_i = 1 & \end{array}$$

Resolving the linear system equation, we have

$$\begin{array}{ll} p_0 = \frac{1-\rho}{1+\rho^{K+2}} & \\ p_k = \frac{\rho^K(1-\rho)}{1+\rho^{K+2}} & 1 \le k \le K+1 \\ p_k = \frac{1-2\rho+\rho^{K+2}}{(1-\rho)(1-2\rho))} & k \ge K+2 \end{array}$$

where $\rho = \frac{\lambda}{\mu}$

4. (a)

$$\begin{array}{ll} \frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t) & \\ \frac{dP_k(t)}{dt} = -(\lambda + 2\mu)P_k(t) + \lambda P_{k-1}(t) + 2\mu P_{k+1}(t) & k > 1. \end{array}$$

(b)

$$\begin{array}{ll} p_k = \rho \frac{\rho}{2}^{k-1} p_0 & k > 0. \\ p_0 = \frac{2-\rho}{2+\rho}. & \end{array}$$

where $\rho = \frac{\lambda}{\mu}$

5. (a)

$$
\begin{aligned}
p_1 &= \rho p_0 \\
p_k &= \alpha^{\frac{k(k-1)}{2}} \rho^k p_0 \quad k \geq 1
\end{aligned}
$$

where $\rho = \frac{\lambda}{\mu}$

(b)

$$
\begin{aligned}
p_0(1 + \sum_{k=1}^{\infty} \alpha^{\frac{k(k-1)}{2}} \rho^k) &= 1 \\
\Rightarrow \quad p_0 &= \frac{1}{1 + \sum_{k=1}^{\infty} \alpha^{\frac{k(k-1)}{2}} \rho^k}
\end{aligned}
$$

6. (a)

$$
\begin{aligned}
a(t) &= \lambda q (1 - \lambda q)^{\frac{t}{q} - 1} \qquad t = q, 2q, 3q, \cdots. \\
b(x) &= \sigma^{\frac{x}{q} - 1} (1 - \sigma) \qquad x = q, 2q, 3q, \cdots.
\end{aligned}
$$

(b)

$$
p_k = (1 - \rho)\rho^k \quad k \geq 0
$$

where

$$
\rho = \tfrac{\lambda q}{1 - \sigma}
$$

(c)

$$
\begin{aligned}
\bar{N} &= \sum_{k=0}^{\infty} k p_k \\
&= \sum_{k=0}^{\infty} k \rho^k (1 - \rho) \\
&= \frac{\lambda q}{1 - \sigma - \lambda q}
\end{aligned}
$$