Answers of Homework1

Problem 1: (Conditional Probability)

An insurance company believes that people can be divided into two classes: those who are accident-prone and those who are not. Their statistics show what an accident-prone person will have an accident at some time within a fixed 1 year period with probability .5, whereas this probability decreases to .3 for a non-accident-prone person. If we assume that 40 percent of the population is accident prone,

(a) What is the probability that a new policyholder will have an accident within a year of purchasing a policy? (3 points)

(b) Suppose that a new policyholder has an accident within a year of purchasing a policy. What is the probability that he or she is accident-prone? (3 points)

(c) What is the conditional probability that a new policy holder will have an accident in his or her second year of policy ownership, given that the policyholder has had an accident in the first year? (4 points)

Answer

Define events

A = { person has an accident within 1 year period }

B = { person is accident prone }

(a)

$$P(A) = P(A \cap B) + P(A \cap \overline{B}) = P(A \mid B)P(B) + P(A \mid \overline{B})P(\overline{B})$$

$$= 0.4 \times 0.5 + 0.6 \times 0.3 = 0.38$$

(b)

$$P(B \mid A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \mid B)P(B)}{P(A)} = \frac{0.5(0.4)}{0.38} = 0.53$$

(c) $\dfrac{0.4 \times 0.5 \times 0.5 + 0.6 \times 0.3 \times 0.3}{0.38} = 0.41$

Problem 2: (Discrete State, Discrete Tim Markov Chain)

There is a man who can be found in one of 3 cities, Abra, Zeus or Suscsamad on any day. If he is in city Abra on any day, he will spend the next day in city Zeus with probability 3/4, or will spend the next day in city Sucsamad with probability ¼. If he is in city Zeus on any day, he will spend the next day in city Abra with probability ¼, or will spend the next day in city Sucsamad with probability 3/4. Whenever he spends a day in city Sucsmamad, he will spend the next day in city Abra with probability ¼, or will spend the next day in city Zeus with probability ¼ or stay for another day in city Sucsmad with probability 1/2. All of these decisions made each day, independent of any decisions made in the past.

(a) Draw the fully labeled state diagram corresponding to the Makov-Chain describing the city he is in. (2 points)

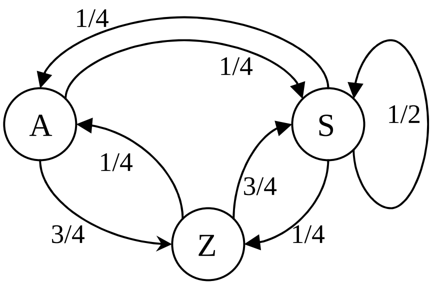
(b) Solve for the equilibrium balance equations for this chain (3 points)

(c) Solve for the equilibrium probabilities in terms of Abra, and Zeus and Suscsmad. (3 points)

(d) Given he is in city Zeus on a particular day, what is the probability that he is in city Zeus exactly 3 days later? (2 points)

Answer

(a) In the graph, respectively, node A, Z, and S represent the probability of the man to stay in Abra, Zeus, and Suscsamad.



(b)

From A city: $P_A = \frac{1}{4}P_Z + \frac{1}{4}P_S$

From Z city: $P_Z = \frac{3}{4}P_A + \frac{1}{4}P_S$

From S city: $P_S = \frac{1}{4}P_A + \frac{3}{4}P_Z + \frac{1}{2}P_S$, and also $P_A + P_Z + P_S = 1$

(c) $P_A = \frac{1}{5}$, $P_Z = \frac{7}{25}$, $P_S = \frac{13}{25}$

(d) $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & \frac{3}{4} & \frac{1}{4} \\ \frac{1}{4} & 0 & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}^3 = \begin{bmatrix} \frac{13}{64} & \frac{1}{4} & \frac{35}{64} \end{bmatrix}$. So, the answer is $\frac{1}{4}$

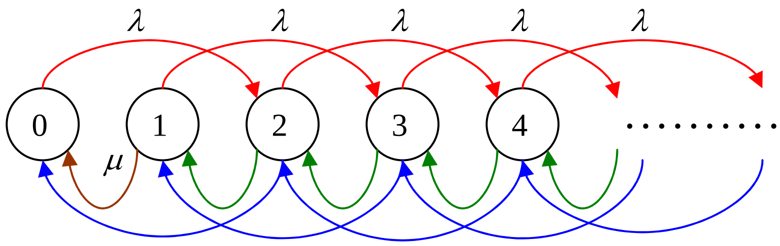
Problem 3: (Discrete State, Continuous Time Markov Chain)

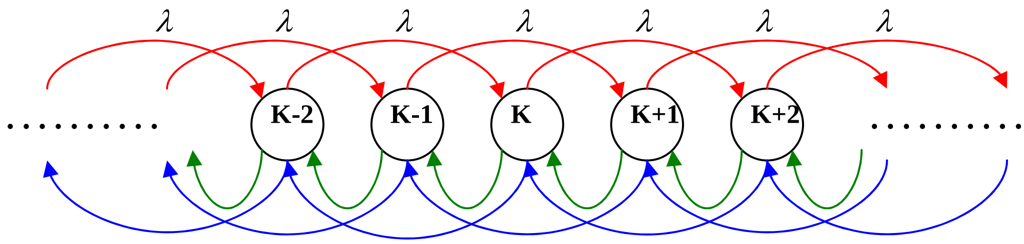
Let us consider an M/M/1 queuing system with bulk arrivals and bulk service. Arrival instants occur at a rate λ, but at each arrival instant, two customers arrive. Service occurs at a rate μ, but at each service completion, the customer who was in service leaves and with probability α, the customer at the head of the queue (if any) will also leave at the same time without ever entering service!

a) Draw a completely labeled state diagram for the number of customers in system (3)
b) Write down the full set of balance equations for the equilibrium state probabilities P_k
c) Find as simple an expression as possible for P(z) in terms of p_0, p_1, α, λ, and μ (3 points)
d) Find a single explicit expression relating p_0 and p_1. (3 points)
e) How would you find another relation between α, λ, and μ? (3 points)
f) Explain why this constraint behaves the way it does for α = 0 and 1. (4 points)

Answer

(a)





The value of green line is $(1-\alpha)\mu$ and the blue one is $\alpha\mu$

(b)

$$\lambda P_0 = \mu P_1 + \alpha\mu P_2$$

$$(\lambda+\mu)P_1 = (1-\alpha)\mu P_2 + \alpha\mu P_3$$

$$(\lambda+\mu)P_k = (1-\alpha)\mu P_{k+1} + \alpha\mu P_{k+2} + \lambda P_{k-2}, k \ge 2$$

(c)

$$Z^0 \Rightarrow \lambda P_0 = \mu P_1 + \alpha\mu P_2$$

$$Z^1 \Rightarrow (\lambda+\mu)P_1 = (1-\alpha)\mu P_2 + \alpha\mu P_3$$

$$Z^k \Rightarrow (\lambda+\mu)P_k = (1-\alpha)\mu P_{k+1} + \alpha\mu P_{k+2} + \lambda P_{k-2}, k \ge 2$$

Sum:

$$\lambda P(Z) + \mu[P(Z) - P_0] = \mu P_1 + \frac{(1-\alpha)\mu}{Z}[P(Z) - P_0 - P_1 Z] + \frac{\alpha\mu}{Z^2}[P(Z) - P_0 - P_1 Z] + Z^2 \lambda P(Z)$$

(d) Use "P(1) = 1" relation in the result of (c). Remember to eliminate the (Z-1) term in the equilibrium.